Problem 1

Evaluate the integral:

$$\boldsymbol{I = \int_0^1 x\, ln(1 + x)\, dx}$$

Solution:

Let u = ln(1 + x), dv = x dx ⇒ du = dx/(1 + x), v = x^2/2.

$$\int x\, ln(1 + x)\, dx = (x^2/2)\, ln(1 + x) - \int (x^2/2)/(1 + x)\, dx$$

Simplify using polynomial division and compute the definite integral:

$$I = [(x^2/2)\, ln(1 + x)]_0^1 - 1/2\, [x^2/2 - x + ln(1 + x)]_0^1 = 1/4$$

Answer: I = 1/4.

Problem 2

Solve the initial value problem:

$$\boldsymbol{dy/dx = y - x,\ \ y(0) = 1}$$

Solution:

Rewrite as dy/dx - y = -x and find the integrating factor μ(x) = e^{-x}.

$$d/dx(e\text{^}\{-x\}\,y) = -x\,e\text{^}\{-x\}$$

Integrate both sides and apply the initial condition:

$$y = x - 1 + 2\,e^{x}$$

Answer: $y = x - 1 + 2e^{x}$.

Problem 3

Let f(x, y, z) = xyz. Compute the gradient ∇f at (1, 2, 3) and the directional derivative along v = (1, -1, 1).

Solution:

Partial derivatives: f_x = yz, f_y = xz, f_z = xy.

$$\nabla f(1,2,3) = (6,3,2)$$

Unit vector along v:

$$u = (1,-1,1)/\sqrt{3}$$

$$Directional\ derivative = \nabla f \cdot u = (6 - 3 + 2)/\sqrt{3} = 5/\sqrt{3}$$

Answer: ∇f(1, 2, 3) = (6, 3, 2); directional derivative = 5/√3.